Lesson 4 - Functions

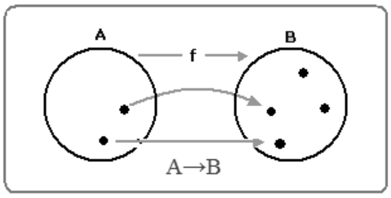
What is a function?

A **function** is a rule that produces a correspondence between the elements of two sets A and B, such that to each element in A there corresponds *one and only one* element in B.

So a function is a particular kind of relation between the sets A and B.

If we look at the figure below we see that for every element a in A there is **exactly one arrow** that starts from a and points to an element of B.

A function

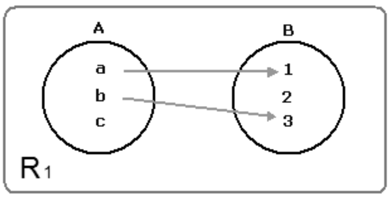


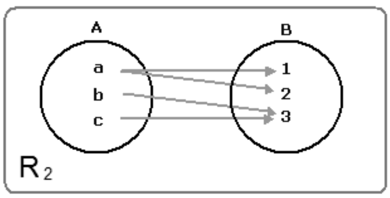
The expression *one and only one* used in the definition of a function implies two conditions:

- **for every** $x \in A$ there exists an element $y \in B$ that corresponds to x
- such element $y \in B$ is **unique**

Examples of relations which are not functions

Consider the two arrow diagrams below, corresponding to relations $\mathcal{R}_1$ and $\mathcal{R}_2$ respectively.





The diagram for $\mathcal{R}_1$ is **not** a function because there is an element in A (namely c) which does not correspond to an element of B.

The diagram for $\mathcal{R}_2$ is **not** a function because there is an element in A (namely a) which corresponds to *more than one* element of B (namely the elements 1 and 2).

Notation

Let A and B be two non-empty sets. A function from A to B is often denoted by a lowercase letter, for example f. We can write this as

$$f : A \to B$$

We can use a similar notation to say that to an element $x \in A$ there corresponds the element $y \in B$ and write

$$x \mapsto y$$

We also say that y is the **image** of x by f.
To denote a function we can also use the notation

$$y = f(x)$$

We call A the **domain** and B the **codomain** of the function f. The **range** is the set of all the images of the elements of A and is thus a subset of B. In mathematical notation the range can be defined as

$$R = \{y \in B \mid \exists x \in A(y = f(x))\}$$

If $y \in B$, we call the set of all elements x in A such that y is the image of x the **pre-image** of y and we use the notation

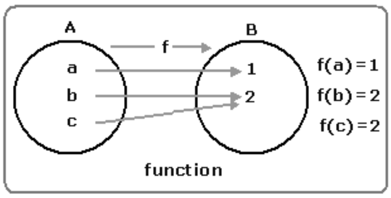
$$f^{-1}(y) = \{x \in A \mid f(x) = y\}$$

If $C \subset A$, the **image** of C is the set of all images of elements of C and we write

$$f(C) = \{y \in B \mid x \in C \wedge y = f(x)\}$$

Example

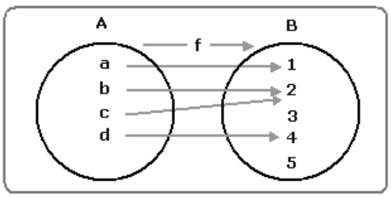
The diagram below represents a function f between the sets $A = \{a, b, c\}$ and $B = \{1, 2\}$.



In this example we see that the element $1 \in B$ is the image of $a \in A$ and we can write $f(a) = 1$, while 2 is the image of two elements of A, namely b and c, so we can write both the relations $f(b) = 2$ and $f(c) = 2$.

Another Example

The diagram below represents a function f between the sets $A = \{a, b, c, d\}$ and $B = \{1, 2, 3, 4, 5\}$. What is the range of f?



The range of f is the set $R = \{1, 2, 4\}$

Injective functions or 1-1 map

A function $f : A \to B$ is **injective**, if every element of B is the image of *at most* one element of A.

In mathematical notation we can write the following relation for f to be injective:

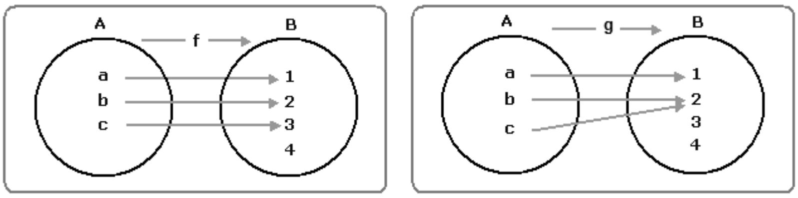
$$\forall x, y \in A[x \neq y \to f(x) \neq f(y)]$$

or equivalently

$$\forall x, y \in A[f(x) = f(y) \to x = y]$$

Example

Consider the two arrow diagrams below, corresponding to functions f and g respectively.



We see that f is an injective function, because every element of B is the image of at most one element of A, while g is not injective because $2 \in B$ is the image of both b and c.

Surjective functions or onto map

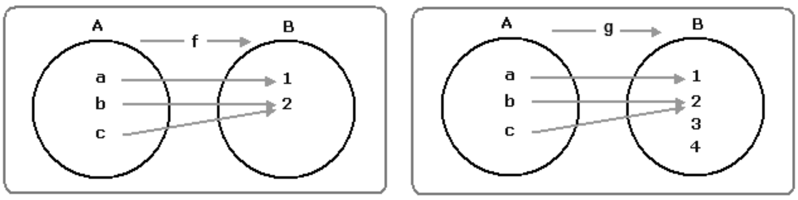
A function $f : A \to B$ is **surjective**, if every element of B is the image of *at least* one element of A.

In mathematical notation we can write the following relation for f to be surjective:

$$\forall y \in B[\exists x \in A(f(x) = y)]$$

Example

Consider the two arrow diagrams below, corresponding to functions f and g respectively.



We see that f is a surjective function, because every element of B is the image of at least one element of A, while g is not surjective because 3, for example, is not the image of any element of A.

Bijective function

A function f that is both injective and surjective is called a **bijective** function.

Exercises

Exercise 1 *Consider the sets* $A = \{1, 2, 3, 4\}$, $B = \{a, b, c, d, e\}$ *and the relation* $f = \{(1, b), (3, d), (2, b), (4, e)\}$. *Draw the arrow representation for* f *and answer to the following questions.*

1. *Is* f *a function?*
2. *Is it injective? Is it surjective?*
3. *Which is the image of 2?*
4. *Which is the pre-image of* b*?*
5. *Which is the pre-image of* c*?*
6. *Which is the image of* $\{1, 3\}$*?*

Exercise 2 *If* $f : A \to B$ *is a function and* A *has 5 elements while* B *has 4 elements, can* f *be injective?*

Exercise 3 *If* $f : A \to B$ *is a function and* A *has 5 elements while* B *has 6 elements, can* f *be surjective?*

Exercise 4 *Given the set* $\mathbb{N}$ *of natural numbers represented in decimal notation, consider the function* $f : \mathbb{N} \to \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$ *that associates to every natural number* n *the rightmost digit in decimal representation. Tell if* f *is injective. Is it surjective?*

Numerical functions

When $f : A \to B$ is a function and A, B are sets of numbers, then we say that f is a **numerical function**.

Sometimes it is possibile to define a numerical function by a formula that tells how to calculate the image of every element in the domain.

Consider for example the function $f : \mathbb{N} \to \mathbb{N}$ defined by $n \mapsto 3n - 2$.

The formula tells that, to obtain the image of a number n, you have to multiply it by 3 and then subtract 2. For example, the image of 4 is 10 because $3 \cdot 4 - 2 = 10$ and we can thus write $f(4) = 10$.

Exercise 5 *For each of the following formulas defining numerical functions in the set* $\mathbb{Q}$ *of rational numbers determine the image of 0, 1,-1. Which formulas define injective functions? and surjective functions?*

1. $x \mapsto x - 1$
2. $x \mapsto x^2$
3. $x \mapsto 3x$
4. $x \mapsto |x + 1|$

Constant functions

A function $f : A \to B$ is a **constant function** if there is a $k \in B$ such that $f(x) = k$ for every $x \in A$.

In other words a function is constant if the range contains only one element.

Identity function

For every set A there is a function $f : A \to A$ such that $f(x) = x$ for every $x \in A$. This function is called identity function on A and is denoted by I_A.