Lesson 3 - Equivalence Relations and Order Relations

In this lesson we want to study two particular cases of relations on a set A: **equivalence relations** and **order relations**.

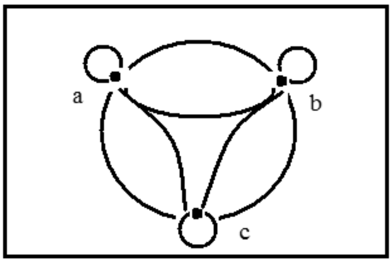
Equivalence relation

We start with the definition:
A relation $\mathcal{R}$ on set A that is reflexive, symmetric and transitive is called an **equivalence relation**.

Example

Consider the set $A = \{a, b, c\}$ and the relation $\mathcal{R}$ in A represented in the graph below.

An equivalence relation



Equivalence Relation

A careful inspection of the graph shows that $\mathcal{R}$ is reflexive, symmetric and transitive and hence an equivalence relation.

Example

Consider the following relation $\mathcal{R}$ defined on the set of integers:

$\mathcal{R}$: "$a - b$ is an integer number"
We can also write

$$\mathcal{R} = \{(a, b)|a - b \in \mathbb{Z}\}$$

We want to show that $\mathcal{R}$ is an equivalence relation.

Solution

We have to show that $\mathcal{R}$ is reflexive, transitive and symmetric.

i.) *Reflexivity*
For every $a \in \mathbb{Z}$ we have $a - a = 0$ and 0 is an integer. This shows that $a\mathcal{R}a$ and hence $\mathcal{R}$ is reflexive.

ii.) *Symmetry*
Now, suppose that $a\mathcal{R}b$ (that is $a - b$ is an integer). We have to show that also $b\mathcal{R}a$. But this means that $b - a$ must be an integer, and this is true because $b - a$ is the opposite of $a - b$. So $\mathcal{R}$ is symmetric.

iii.) *Transitivity*
Now, suppose that $a\mathcal{R}b$ and $b\mathcal{R}c$, that is $a - b$ and $b - c$ are both integers. We have to show that also $a\mathcal{R}c$. But this means that $a - c$ must be an integer, and this is true because we can add the two integers $a - b$ and $b - c$ and we obtain $a - b + b - c = a - c$ which is an integer because the integers are closed with respect to addition. So $\mathcal{R}$ is transitive.
i) ii) and iii) implies that $\mathcal{R}$ is an equivalence relation.

Example

Consider the relation $\mathcal{R}$: "*a is a brother of b*" defined on a set of people A. Verify if it is reflexive, symmetric and transitive and decide if it is an equivalence relation.

i.) *Reflexivity*

$\mathcal{R}$ is not reflexive because this would mean that a person is a brother of himself which is not true.

ii.) *Symmetry*

Now, suppose that $a\mathcal{R}b$, that is a is a brother of b. Does it follow that $b\mathcal{R}a$, that is b is a brother of a? Well, it depends on the set of persons, because if a is male and b is female the answer is no (b is a sister of a). So $\mathcal{R}$ is, in general, not symmetric.

iii.) *Transitivity*

Now, suppose that $a\mathcal{R}b$ and $b\mathcal{R}c$, that is a is a brother of b and b is a brother of c. If a, b, c are all different then we can say that a is a brother of c, thus proving transitivity.

Remark: In the definition of transitivity we do not require a, b and c to be distinct. So we must check the special case in which c is equal to a: if a is a brother of b, and b is a brother of a does it follow that a is a brother of a? The answer is no, and so, in general, $\mathcal{R}$ is not transitive. This is a strange example and the problem lies in the fact that the relation in not reflexive. Intuitively we would consider $\mathcal{R}$ to be transitive, but according to the definition we cannot say it is transitive. Note that some textbooks say that this relation is transitive (and maybe also reflexive).

Because reflexivity always fails, $\mathcal{R}$ is NOT an equivalence relation.

An important example - Congruence modulo n - Exercise

Consider a fixed non-negative integer n. Define the following relation $\mathcal{R}$ on the set $\mathbb{Z}$ of integers: "*x and y have the same remainder when divided by n*". Show that $\mathcal{R}$ is an equivalence relation on $\mathbb{Z}$.

This equivalence relation is very important and it is called **congruence modulo** n. In other words we say that two integers x and y are congruent modulo n if $x - y$ is a multiple on n; in symbols we write $x \equiv_n y$.

Exercise 1 *Let $A = \{$ members of a family $\}$. The relation $\mathcal{R}$ is "a is the father of b". Show that the relation $\mathcal{R}$ does not satisfy any property defining an equivalence relation.*

Exercise 2 *If A is the set of students of $1°B$, tell which of the following are equivalence relations on A:*

a) x's zodiac sign is the same as y's

b) x is taller than y

c) x's eyes color is the same as y's

Exercise 3 *Given the set A of all lines in cartesian plane, tell which of the following are equivalence relations on A:*

a) x and y have a common point

b) x and y are perpendicular

c) x and y are parallel

Exercise 4 *Given the set $\mathbb{N}$ of natural numbers represented in decimal notation, consider the relation $\mathcal{R}$: "x has at least one digit in common with y". Tell if $\mathcal{R}$ is an equivalence relation.*

Equivalence relations and partitions of a set

The power of an equivalence relation lies in its ability to partition a set into the disjoint union of subsets called **equivalence classes**. Because of its power to partition a set, an equivalence relation is one of the most used and pervasive tools in mathematics.

Example

Consider a set A of colored objects and the relation $\mathcal{R}$: "*x has the same color as y*". This is clearly an equivalence relation (verify it!). All the objects are subdivided according to their *color*. The subsets so obtained are

i) all non empty

ii) disjoint in pairs

iii) their union is the set A

But this defines a partition of the set A.

Each subset is called an **equivalence class**.
The set of all equivalent classes is called the **quotient set** of the relation. It is a set of subsets of A.
In this example an equivalence class is simply a *color* and the quotient set is the set of *colors* of the objects in A.

In moving from the set A to the quotient set relative to $\mathcal{R}$ we use an **abstraction**, that is we forget all the properties of the objects in A and we consider only the color.

Exercise 5 *Find the quotient set of the following equivalence relations:*

$\mathcal{R}_1$: "*x and y live in the same city*" in the set of students of your school

$\mathcal{R}_2$: "*x and y have the rightmost digit equal*" in the set natural numbers

$\mathcal{R}_3$: "*x and y have the same number of edges*" in the set of polygons in the plane

$\mathcal{R}_4$: "*x and y play in the same team*" in the set of professional (serie A) soccer players in Italy

$\mathcal{R}_5$: "*x and y are congruent modulo 5*" in the set of integers

Two other properties of relations

We now want to study two new properties of relations. As usual, $\mathcal{R}$ denotes a relation in A.

Antireflexivity

A relation is **antireflexive (or irreflexive)** when all elements in the set A are not related to themselves.

More formally we can say that

A relation $\mathcal{R}$ on set A is **antireflexive** when for all a in A, a is not $\mathcal{R}$-related to itself or in mathematical notation

$$\forall a \in A, a \not\mathcal{R} a$$

Example

The relation $\mathcal{R}$: "*a is younger than b*" defined on a set of persons is antireflexive because we cannot say that a person is younger than himself.

Antisymmetry

A relation $\mathcal{R}$ on a set A is **antisymmetric** when, for every element a which is related to b and $a \neq b$, it happens that b is not related to a. In mathematical notation, we can write:

$$\forall a, b \in A (a \neq b \wedge a\mathcal{R}b \Rightarrow b \not\mathcal{R} a)$$

Example

The relation $\mathcal{R}$: "*a is younger than b*" defined in the preceding example is also antisymmetric because if a person a is younger than b then b is not younger than a.

Order relations

We now define two basic types of order relations.

Non-strict order relation

A relation in a set A is a **non-strict order relation** if it is reflexive, antisymmetric and transitive.

Example

Consider the relation $\mathcal{R}$: "*x is a multiple of y*" defined on the set $\mathbb{N}$ of natural numbers.

We can easily verify that $\mathcal{R}$ is reflexive, antisymmetric and transitive and so it is a non-strict order relation.

Strict order relation

A relation in a set A is a **strict order relation** if it is antireflexive, antisymmetric and transitive.

Example

Consider the relation $\mathcal{R}$: "$x < y$" defined on the set $\mathbb{N}$ of natural numbers.

We can easily verify that $\mathcal{R}$ is antireflexive, antisymmetric and transitive and so it is a strict order relation.

Partial Order and Total Order

We give the following important definition:

An order relation $\mathcal{R}$ (strict or non-strict) on a set A is a **total order** if, for every pair of distinct elements a and b in A, it is true that a is in relation with b or b is in relation with a.

An order relation $\mathcal{R}$ (strict or non-strict) on a set A which is not a total order is called a **partial order**.

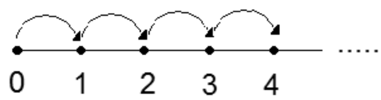
Example

Consider the relation $\mathcal{R}$: "$x < y$" defined on the set $\mathbb{N}$ of natural numbers.

We saw before that it is an order relation. Now we ask: is it a total order? The answer is yes: if we consider two distinct natural numbers n and m, one of the relations $n < m$ and $m < n$ is true.

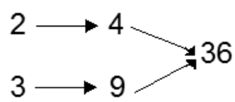
If we have a total order on a set A then the elements of A can be arranged in a chain as shown in the figure below for the relation $\mathcal{R}$: "$x < y$" in the last example.

A total order in $\mathbb{N}$



If we have an order relation $\mathcal{R}$ on a set A which is not a total order, then A cannot be arranged in a chain because there must be two distinct elements a and b such that neither $a\mathcal{R}b$ nor $b\mathcal{R}a$ is true. In the figure below is represented the relation $\mathcal{R}$: "*x is a divisor of y*" in the set $A = \{2, 3, 4, 9, 36\}$.

A partial order not total



We can see that 2 is not a divisor of 3 and 3 is not a divisor of 2 so the order is not total.

Exercise 6 *Let A be a set and consider the relation $\mathcal{R}$: "X is a subset of Y" defined in $P(A)$ (the power set of A). Show that the relation $\mathcal{R}$ is an order relation. Is it strict? Is it a total order?*

Exercise 7 *Let $\mathbb{Q}$ be the set of rational numbers and consider the relation $\mathcal{R}$: "$|x| \leq |y|$". Is it an order relation?*

Exercise 8 *Which of the following are order relations on $\mathbb{N}$?*

$\mathcal{R}_1$: *x is two times y*

$\mathcal{R}_2$: *x is the successor of y*

$\mathcal{R}_3$: *x has less zeros in its decimal representation than y*