Lesson 2 - Relations on a set and their properties

Relations on a set

In this lesson we consider a particular case of relations: relations in which the second set (target set) B coincides with A, so $\mathcal{R} \subset A \times A$.

In this case we say that $\mathcal{R}$ is a relation **defined** in A (or simply in A).

We now want to study some interesting properties of relations. From now on, $\mathcal{R}$ is a relation in A.

Reflexivity

A relation is **reflexive** when all elements in the set A are related to themselves.

More formally we can say that

A relation $\mathcal{R}$ on set A is **reflexive** when for all a in A, a is $\mathcal{R}$-related to itself or in mathematical notation

$$\forall a \in A, a\mathcal{R}a$$

Graphs

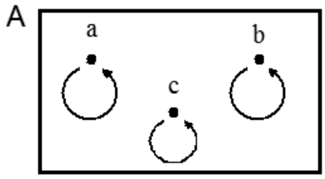
If we are given a relation on a set A we have an alternative way to represent it by **graphs**.

Simply draw a point for each element in A, which we call a **node**. For every pair of elements in relation, draw an arrow starting from the first element and pointing to the second. If an element is in relation with itself the arrow starts and ends in the same node.

Example

Consider the set $A = \{a, b, c\}$ and the relation $\mathcal{R}_1$ in A represented in the first graph below.

A reflexive relation

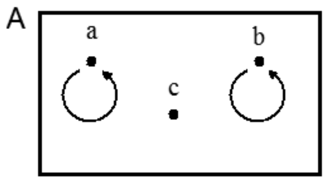


Can we tell, simply by analyzing the graph, that $\mathcal{R}_1$ is a reflexive relation?

YES, because we easily see that each element x of A has an arrow that points to itself and this means that $x\mathcal{R}_1x$.

Consider now another relation $\mathcal{R}_2$ in A with the graph represented below.

A non reflexive relation



This time the relation $\mathcal{R}_2$ is non-reflexive because the element c is not related to itself.

Another example

Consider the set $A = \{1, 2, 3\}$.

Then, one of the possible reflexive relations on A is:

$$\mathcal{R}_1 = \{(1,1), (2,2), (3,3), (1,2), (1,3)\}$$

It is reflexive because it includes $(1,1)$, $(2,2)$ and $(3,3)$, that is **all** possible pairs formed with members of A, with the first and second element equal.

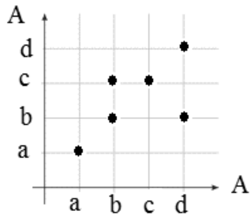
However, the following is not a reflexive relation:

$$\mathcal{R}_2 = \{(1,1), (2,2), (1,2), (1,3)\}$$

It is not a reflexive relation because the ordered pair $(3,3)$ is **missing** and this violates the condition that every element of the set A is related to itself.

A worked example

We have a set $A = \{a, b, c, d\}$ and a relation $\mathcal{R}$ on A represented in Cartesian plane form, shown in figure below. Decide whether the relation is reflexive or not.



Solution

In the graph a subset of $A \times A$ is shown. The subset from the Cartesian plane is:

$$\{(a,a), (b,b), (b,d), (c,b), (c,c), (d,d)\} \subset A \times A$$

In this subset we find **all** the ordered pairs $(a,a), (b,b), (c,c), (d,d)$. So, the relation is reflexive.

Exercise 1 *Given the set* $A = \{x \in \mathbb{N} | \ 3 \leq x \leq 8\}$ *, represent the relation*
$\mathcal{R}$: "$x + y$ *is an even number*"
in Cartesian plane and tell if it is reflexive.

Exercise 2 *Given the set* $A = \{x \in \mathbb{Z} | \ -2 \leq x \leq 2\}$ *, represent the relation*
$\mathcal{R}$: "$x \cdot y > 0$"
in Cartesian plane and tell if it is reflexive.

Exercise 3 *Given the set* $A = \{a, b, c, d, e\}$*, construct the graph of a reflexive relation* $\mathcal{R}_1$ *such that there are exactly six arrows. Does there exist a reflexive relation* $\mathcal{R}_2$ *on A with less than 5 arrows?*

Exercise 4 *Given the set* $A = \{a, b, c\}$ *and the relation* $\mathcal{R} = \{(a,b), (b,a), (b,c)\}$ *on A, how can we find the smallest reflexive relation* $\mathcal{R}_1$ *containing* $\mathcal{R}$*?*

Symmetry

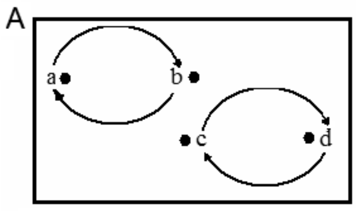
A relation $\mathcal{R}$ on a set A is **symmetric** when, for every element a which is related to b, it also happens that b is related to a. In mathematical notation, we can write:

$$\forall a, b \in A, a\mathcal{R}b \Rightarrow b\mathcal{R}a$$

Example

Consider the set $A = \{a, b, c, d\}$ and the relation $\mathcal{R}_1$ in A represented in the first graph below.

A symmetric relation

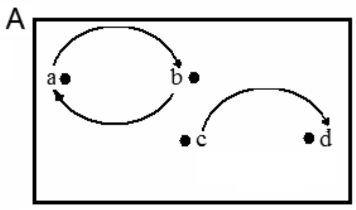


Can we tell simply by analyzing the graph that $\mathcal{R}_1$ is a symmetric relation?

YES, because we easily see that for every arrow between two elements of A there is also the arrow between the same elements with reversed direction.

Consider now the relation $\mathcal{R}_2$ in A with the graph represented below.

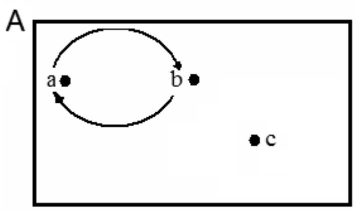
A non symmetric relation



In this case the relation $\mathcal{R}_2$ is non-symmetric because, for the arrow corresponding to the ordered pair (c, d), there is not the reverse arrow corresponding to the ordered pair (d, c).

Consider another relation $\mathcal{R}_3$ in A with the graph represented below.

Another symmetric relation



The relation $\mathcal{R}_3$ is symmetric. There are no outward arrows from c, so there is nothing to check about c.

An important observation

In a symmetric relation every instance of a relation has a **mirror image**. This means that if, for example, $(1, 3)$ is an instance, then $(3, 1)$ is also an instance in the relation. Clearly, ordered pairs of equal elements like $(1, 1)$ or $(2, 2)$ are themselves their own mirror images.

A worked example

We have a set $B = \{1, 2, 3, 4\}$ and a relation $\mathcal{R}$ defined in $B \times B$:

$$\{(1, 1), (2, 3), (4, 3), (3, 2), (2, 2), (3, 3)\} \subset B \times B$$

Decide whether the relation $\mathcal{R}$ is symmetric or not.

Solution

It is not symmetric because the ordered pair $(4, 3)$ belongs to $\mathcal{R}$, but $(3, 4)$ is missing. You can also check that $(4, 4)$ is missing.

Some exercises

Exercise 5 *Given the set $A = \{0, 1, 2, 3\}$, tell which of the following relations are symmetric relations in A.*

$$\mathcal{R}_1 = \{(0, 0), (1, 1), (2, 2), (3, 3), (4, 4)\}$$

$$\mathcal{R}_2 = \{(0, 1), (0, 2), (0, 3), (1, 2), (2, 3)\}$$

$$\mathcal{R}_3 = \{(1, 2), (2, 1), (3, 2), (2, 3)\}$$

$$\mathcal{R}_4 = \{(2, 1), (3, 0)\}$$

$$\mathcal{R}_5 = \{(1, 1), (2, 0), (0, 2)\}$$

Exercise 6 *Given the set $A = \{0, 1, 2, 3, 4, 5\}$, which of the following relations are symmetric?*

$\mathcal{R}_1$: “$x + y = 4$“
$\mathcal{R}_2$: “$x = 2y$“
$\mathcal{R}_3$: “$x \leq y$“

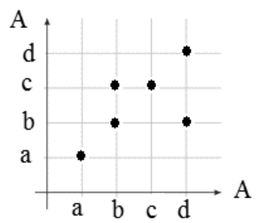
Exercise 7 *Given the set $A = \{0, 1, 2, 3, 4\}$, which of the following relations are symmetric?*

$\mathcal{R}_1$: “$x^2 + y^2 = 4$“
$\mathcal{R}_2$: “$x^2 = y^2$“
$\mathcal{R}_3$: “$x + y = 0$“

Exercise 8 *Given the set $A = \{a, b, c, d, e\}$, construct the graph of a symmetric relation $\mathcal{R}_1$ such that there are exactly four arrows. Does there exist a symmetric relation $\mathcal{R}_2$ on A with 1 arrow?*

Exercise 9 *Given the set $A = \{a, b, c\}$ and the relation $\mathcal{R} = \{(a, b), (b, a), (b, c)\}$ on A, how can we find the smallest symmetric relation $\mathcal{R}_1$ containing $\mathcal{R}$?*

Exercise 10 *Given the graph of the relation in the figure below, complete it to obtain a symmetric relation.*



Transitivity

A binary relation $\mathcal{R}$ on a set A is **transitive** if, whenever an element a is related to an element b, and b is in turn related to an element c, then a is also related to c.

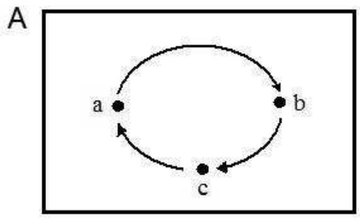
In mathematical notation we can write:

$$\forall a, b, c \in A, a\mathcal{R}b \wedge b\mathcal{R}c \Rightarrow a\mathcal{R}c$$

Example

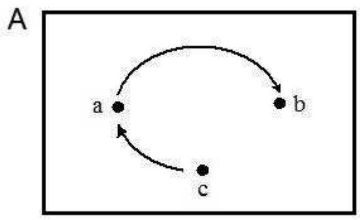
Consider the set $A = \{a, b, c\}$ and the relation $\mathcal{R}_1$ in A represented in the first graph below.

A transitive relation



$\mathcal{R}_1$ is clearly a transitive relation. But if we remove just one of the arrows, as in the graph below, we obtain a relation which is not transitive.

A non transitive relation



An important observation

If we represent a transitive relation with a graph and there are two arrows connecting three elements, the first with the second and the second with the third, then there **must be** also the arrow connecting the first with the third *directly*.

A worked example

Given the set $A = \{0, 1, 2, 3\}$, decide if the following relation is transitive or not.

$$\mathcal{R} = \{(0,0), (1,1), (2,2), (3,0), (0,2), (0,3), (1,2), (2,3)\}$$

Solution:

It is not a transitive relation because in $\mathcal{R}$ there are the pairs $(3, 0)$ and $(0, 2)$ but there is not the ordered pair $(3, 2)$.

Exercise 11 *Given the set $A = \{1, 2, 3, 4\}$, which of the following relations on A are transitive?*

$\mathcal{R}_1$: "*x is a multiple of y*"
$\mathcal{R}_2$: "*x is a factor of y*"
$\mathcal{R}_3$: "*x is less than y*"

Exercise 12 *Write the properties of each of the following relations in $A = \mathbb{N}$.*

$\mathcal{R}_1$: "*x is two times y*"
$\mathcal{R}_2$: "*x is the successor of y*"
$\mathcal{R}_3$: "*x, y are both divisors of the same number*"

Exercise 13 *Write the properties of each of the following relations in $A = \mathbb{N}$.*

$\mathcal{R}_1$: "$x < y$"
$\mathcal{R}_2$: "$x \leq y$"
$\mathcal{R}_3$: "*x, y have a common divisor different from 1*

Exercise 14 *Which are the properties of the relation in the figure below?*