Lesson 1 - Relations and Functions

What is a relation?

We are given two sets A and B. For example $A = \{Ann, Mary, Lucy\}$ and $B = \{John, Ted, Jane\}$.

We can form the Cartesian Product

$$\begin{aligned} A \times B = \{&(Ann, John), (Ann, Ted), (Ann, Jane),\\ &(Mary, John), (Mary, Ted), (Mary, Jane),\\ &(Lucy, John), (Lucy, Ted), (Lucy, Jane)\} \end{aligned}$$

Now consider the open sentence p : '*a* <u>is the mother of</u> *b*'.

We can think of a as an arbitrary member of the set A and of b as an arbitrary member of the set B.

Let's say that

1. *Ann* <u>is the mother of</u> *Ted*
2. *Mary* <u>is the mother of</u> *John*
3. *Lucy* <u>is the mother of</u> *Jane*

We now ask: Which ORDERED PAIRS of the cartesian product $A \times B$ satisfy the open sentence p?

The answer is simple: just transform into pairs the sentences 1. to 3. and list them in a set which we call $\mathcal{R}$.

$$\mathcal{R} = \{(Ann, Ted), (Mary, John), (Lucy, Jane)\}$$

We have established a **RELATION** between the sets A and B. We are now ready for the formal definition:

FORMAL DEFINITION

A **relation** between the two sets A and B is an arbitrary subset $\mathcal{R}$ of their cartesian product $A \times B$.

In symbols we can write $\quad \mathcal{R} \subset A \times B$

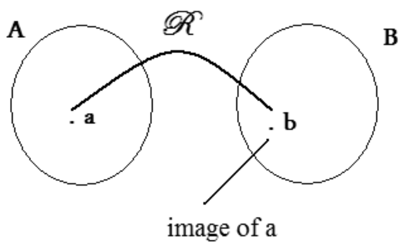
A bit of notation and terminology ....

Suppose now that $\mathcal{R}$ is a relation between A and B.

If $\quad (a, b) \in \mathcal{R} \quad$ we say that $\quad$ *a is in relation with b*

and we can write $\quad (a, b) \in \mathcal{R} \quad$ or $\quad a\mathcal{R}b$

We also say that b is the **image** of a.



A worked example

Consider the following two sets $A = \{5, 10, 20, 30\}$ and $B = \{25, 42, 63, 72\}$. We interpret A as the set of ages of group of persons and B as the set of their weights.

We consider the open sentence p : 'the person of age x has an average weight y'

This sentence defines a **relation** between the two sets of data A and B which we call, as usual, $\mathcal{R}$. It is reasonable to think that, as age grows also weight grows, so in this case

$$\mathcal{R} = \{(5, 25), (10, 42), (20, 63), (30, 72)\}$$

Important!

Notice that $(10, 42) \neq (42, 10)$

This is clearly the case as the ordered pair (10, 42) provides the correct relation between age and weight, i.e., at the age of 10 years the weight of the person is 42 kg.

On the other hand, the ordered pair (42, 10) would indicate that at the age of 42 years the weight of the person is 10kg!

Try to answer to the following questions:

1) Which is the image of 10 relative to $\mathcal{R}$?

2) Is it true or false that 63 is the image of 20?

3) Is it true that 5 is in relation with 42?

4) Is it true that the relation $\mathcal{R}$ is a proper subset of $A \times B$?

5) Is it true that if I add the ordered pair $(35, 75)$ to the relation $\mathcal{R}$, I obtain another relation between the same sets A and B?

6) Is it true that the set of all the images of $\mathcal{R}$ is B?

Some exercises

Exercise 1 *Given the sets* $A = \{0, 1, 2, 3\}$ *and* $B = \{1, 2, 3\}$*, tell which of the following sets are relations in* $B \times A$*.*

$$\mathcal{R}_1 = \{(0, 0), (1, 1), (2, 2), (3, 3), (4, 4)\}$$

$$\mathcal{R}_2 = \{(0, 1), (0, 2), (0, 3), (1, 2), (2, 3)\}$$

$$\mathcal{R}_3 = \{(1, 2), (2, 1), (3, 2)\}$$

$$\mathcal{R}_4 = \{(2, 1), (3, 0)\}$$

$$\mathcal{R}_5 = \{(1, 1), (2, 0), (0, 2)\}$$

Exercise 2 *Given the sets* $A = \{0, 1, 2, 3, 4, 5\}$ *and* $B = \{-1, 0, 1, 2\}$*, list the elements of the following relations:* $\mathcal{R}_1$: "$x + y = 4$" $\mathcal{R}_2$: "$x = 2y$" $\mathcal{R}_3$: "$x \leq y$"

Exercise 3 *Given the sets* $A = \{0, 1, 2, 3, 4\}$ *and* $B = \{-2, -1, 0, 1, 2, 3\}$*, list the elements of the following relations:* $\mathcal{R}_1$: "$x^2 + y^2 = 4$" $\mathcal{R}_2$: "$x^2 = y^2$" $\mathcal{R}_3$: "$x + y = 0$"

Domain and Codomain (Range)

We now give the following important definition. Consider a relation $\mathcal{R}$ defined in $A \times B$.

The set of all first elements of the ordered pairs in $\mathcal{R}$ is called the **domain** of the relation and is referred to as the **independent variable**.

The set of all second elements in $\mathcal{R}$ is called the **codomain** (**range**) and is referred to as the **dependent variable**.

Example

Determine the domain and range for each of the following two relations:

$$\mathcal{R}_1 = \{(0,0),(1,1),(2,4),(3,9),(4,16),(5,25)\}$$

$$\mathcal{R}_1 = \{(-3,4),(-1,0),(2,-2),(-2,2)\}$$

Solution:

$\mathcal{R}_1$ Domain $= \{0,1,2,3,4,5\}$ Range$= \{0,1,4,9,16,25\}$

$\mathcal{R}_2$ Domain$= \{-3,-1,2,-2\}$ Range$= \{4,0,-2,2\}$

Another example

Consider the two sets given below:

$$A = \{2,3,4,5,6\} \qquad B = \{4,6,8,10,12\}$$

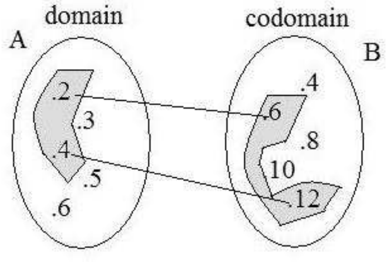
The relationship between them is defined by the proposition: *b is three times a* (b is an element in B and a an element in A).

Determine the domain and range of the relation.

Solution:

Domain: $\{2,4\}$

Range: $\{6,12\}$



Exercise 4 *Given the sets* $A = \{1,2,3,4\}$ *and* $B = \{2,4,6,8\}$*, determine domain and range of the following relations:*

$\mathcal{R}_1$: “*x is a multiple of y*“

$\mathcal{R}_2$: “*x is a factor of y*“

$\mathcal{R}_3$: “*x is the square of y*“

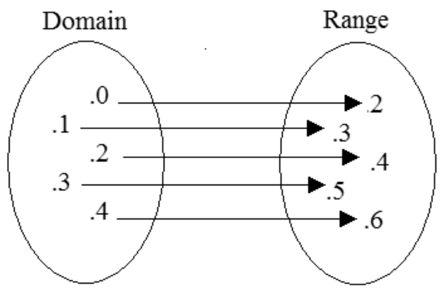
How to represent a relation?

The basic ways of representing relations are **mapping diagrams** and **cartesian plane**.

Mapping diagrams

A mapping diagram as shown below can be used to graph the relation

$$\mathcal{R} = \{(0,2),(1,3),(2,4),(3,5),(4,6)\}$$

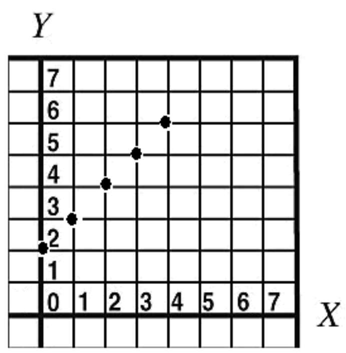


In this kind of representation every arrow starts from a first element of an ordered pair of a relation and has as target the corresponding second element of the same pair.

Cartesian plane

The same relation can be represented in a cartesian plane as shown in the diagram below.

$$\mathcal{R} = \{(0,2),(1,3),(2,4),(3,5),(4,6)\}$$



The idea is to display the elements in the domain in the horizontal axis (X-axis), the elements in the codomain in the vertical axis (Y-axis) and to draw a point (x, y) with X-coordinate x and Y-coordinate y if and only if the pair $(x, y) \in \mathcal{R}$.

Exercise 5 *Represent with mapping diagrams and in cartesian plane the relations of exercise 4.*

Exercise 6 *Represent with mapping diagrams and in cartesian plane the relation*
$\mathcal{R} =$ *'the product $x \cdot y$ is positive'*
defined in $A \times A$ where $A = \{x \in \mathbb{Z} | -2 \leq x \leq 2\}$.

A project

The aim of this project is to collect some data about musical tastes of the students of your class and to represent that data as a relation using cartesian plane.

We are interested in the following two relations:

$\mathcal{R}_1$: the artist y is the preferred artist of student x
$\mathcal{R}_2$: the student x loves musical genre y

Hint: You can proceed as follows. Ask your school mates about their preferred artist and two musical genre they particularly like and then organize all collected data .....

Enjoy yourself with some calculations...

Consider the following formulas (where $n \in \mathbb{N}$)

F1) $n^2 - 1$

F2) $\frac{n-1}{n+1}$

F3) $\frac{n^2-1}{n+1}$

F4) $(-1)^n(n+3)$

F5) $n+5$

F6) $(n+1)(n-1)$

and the following sequences of natural numbers

S1) $-1, 0, \frac{1}{3}, \frac{1}{2}, \frac{3}{4}....$

S2) $5, 6, 7, 8, 9, ...$

S3) $-1, 0, 3, 8, 15....$

S4) $-1, 0, 1, 2, 3, ...$

S5) $3, -4, 5, -6, 7, ...$

S6) $-1, 0, 3, 4, 7....$

Can you determine the relation $\mathcal{R}$ corresponding to the sentence *'the formula* F *generate the sequence* S'?