Lesson 5 - More on Functions

Inverse Function

If $f : A \to B$ is a function from A to B then an **inverse function** for f is a function in the opposite direction, from B to A. If an input x into the function f produces an output y, then inputting y into the inverse function produces the output x. We saw that every relation has an inverse but not every function has an inverse; those that do are called **invertible**.

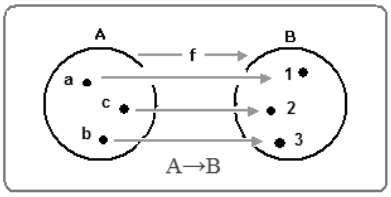
A necessary and sufficient condition for a function f to be invertible is that it is **bijective**.

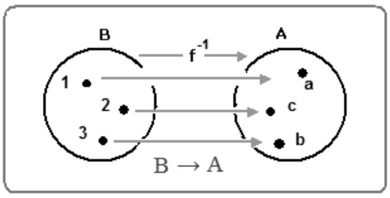
We denote the inverse function of f with the symbol f^{-1} (read f inverse, not to be confused with exponentiation).

The definition of inverse function is the following:

If $f : A \to B$ is a bijective function such that for every $x \in A$ the image of x is $y = f(x)$ then the **inverse** of f is the bijective function $f^{-1} : B \to A$ such that for every $y \in B$ the image of y is $x = f^{-1}(y)$.

A function and its inverse





In the figure above we see that $f(a) = 1$ and $f^{-1}(1) = a$, while $f(b) = 3$ and $f^{-1}(3) = b$.

Remark and example

If a function $f : A \to B$ is injective, is it always possible to find an inverse function for f if we define the domain of f^{-1} as the set of all the images of elements of A, that is the range of f.

For example, if $f : \mathbb{N} \to \mathbb{N}$ is the function that sends the natural number n to two times n, that is $f(n) = 2n$, then f is injective and not surjective. But if we consider the set P of all even numbers (which is the range of f), then $f^{-1} : P \to \mathbb{N}$ defined by $f^{-1}(n) = \frac{n}{2}$ is an inverse for f.

Exercise 1 *Consider the sets* $A = \{0, 1, 2, 3, 4\}$ *and* $B = \{a, b, c, d, e\}$ *and the function* f *such that*

$$0 \mapsto b \qquad 1 \mapsto c \qquad 2 \mapsto a \qquad 3 \mapsto e$$

. Which element must be associated to 4 for f to be an invertible function?

Exercise 2 *Consider the following functions whose domain is* $\mathbb{Z}$ *and tell which of them are invertible.*

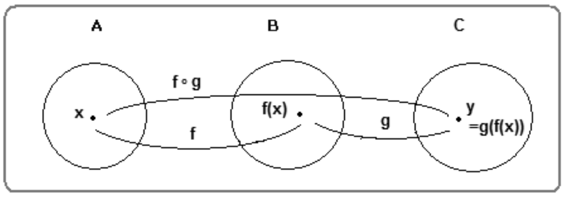
a) $n \mapsto n^2$

b) $n \mapsto n + 1$

c) $n \mapsto -n$

Composition of functions

A composite function represents the application of one function to the results of another. For instance, the functions $f : A \to B$ and $g : B \to C$ can be composed by first computing a $f(x)$ and then applying a function g to the output of $f(x)$. This is shown in the diagram below



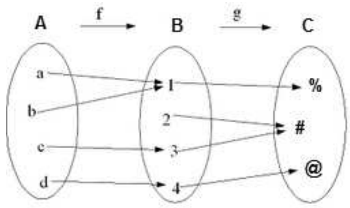
DEFINITION

Given two functions $f : A \to B$ and $g : B \to C$, such that the codomain of the first is equal to the domain of the second we define the **composition** of f with g as the function $g \circ f : A \to C$ such that

$$(g \circ f)(x) = g(f(x))$$

Example

Consider the functions represented in the diagram below



We see that $f(a) = 1$ and $g(1) = \#$.
According to the definition we have also $(g \circ f)(a) = g(f(a)) = g(1) = \#$.

Exercise 3 *Complete the following calculations for the composite function $(g \circ f)$ of the preceding example.*

$(g \circ f)(b) =$

$(g \circ f)(c) =$

$(g \circ f)(d) =$

Exercise 4 *Given the functions $f : \mathbb{Z} \to \mathbb{Z}$ and $g : \mathbb{Z} \to \mathbb{Z}$ defined respectively by $n \mapsto n+1$ and $n \mapsto n+4$ calculate the following values*

$$f(g(3)) = \qquad g(f(-2)) =$$

Some remarks on composition and inverses of functions

The following are important remarks on composition and inverses of functions.

Equality of functions

Two functions $f : A \to B$ and $g : C \to D$ are considered to be equal if they have the same domain and codomain, that is $A = C$ and $B = D$, and for all $x \in A$ we have $f(x) = g(x)$.

Composition is not commutative

If we are given two functions $f : A \to A$ and $g : A \to A$ with the same domain and codomain the we can form both composite functions $(g \circ f)$ and $(f \circ g)$. It is important to note that **in general**

$$g \circ f \neq f \circ g$$

as the following example shows.

Example

Given the functions $f : \mathbb{Z} \to \mathbb{Z}$ and $g : \mathbb{Z} \to \mathbb{Z}$ defined respectively by $n \mapsto n + 1$ and $n \mapsto n^2$ we have

$$(g \circ f)(1) = g(f(1)) = g(2) = 4 \qquad (f \circ g)(1) = f(g(1)) = f(1) = 2$$

so $(g \circ f)(1) \neq (f \circ g)(1)$ and according to our definition $g \circ f \neq f \circ g$.

Identity function is a neuter element for composition

In Lesson 4 we defined the identity function I_A for every set A (remember that for all $x \in A$ we have $I_A(x) = x$). If we are given a function $f : A \to B$ we can consider the following two composite functions

$f \circ I_A : A \to B$

$I_B \circ f : A \to B$

By definition of composition we have

$$(f \circ I_A)(x) = f(I_A(x)) = f(x)$$

so $f \circ I_A = f$ (see also the definition of equality of functions). Moreover

$$(I_B \circ f)(x) = I_B(f(x)) = f(x)$$

and $I_B \circ f = f$. This means that the identity function is a neuter element for the operation $\circ$ of composition of functions.

Composing a function with its inverse

If we are given a bijective function $f : A \to B$ we can consider its inverse $f^{-1} : B \to A$ and form the two composite functions $f^{-1} \circ f : A \to A$ and $f \circ f^{-1} : B \to B$. According to the definitions of composition and inverse of a function we have

$$(f^{-1} \circ f)(x) = f^{-1}(f(x)) = x$$

for all $x \in A$ that is $f^{-1} \circ f$ is the identity I_A on A. Analogously $(f \circ f^{-1})(y) = f(f^{-1}(y)) = y$ for all $y \in B$ that is $f \circ f^{-1}$ is the identity I_B on B.

A worked example

Given the functions $f : \mathbb{Z} \to \mathbb{Z}$ and $g : \mathbb{Z} \to \mathbb{Z}$ defined respectively by $n \mapsto n^2 + 3$ and $n \mapsto n + 1$ we want to find the analytic expression for the two composite functions $f \circ g$ and $g \circ f$.

Solution

$$(f \circ g)(n) = f(g(n)) = f(n + 1) = (n + 1)^2 + 3 = n^2 + 2n + 4$$

$$(g \circ f)(n) = g(f(n)) = g(n^2 + 3) = n^2 + 3 + 1 = n^2 + 4$$

Existence conditions/natural domain of a function

Sometimes a numerical function can be given by defining a rule that tells how to find the image for an element x without specifying the domain of the function. In this case we have to discover the set of numbers for which the rule applies. This set can be a *proper subset* of the set of all (rational or real) numbers and it is called the **natural domain** of the function.

Example

Consider the function f given by the rule

$$x \mapsto \frac{2}{x - 1}$$

We can certainly calculate $f(x)$ for all numbers $x \neq 1$. For the number 1 we are in trouble with the rule because the denominator of the fraction goes to 0. This means that the natural domain of f is the set $A = \{x \in \mathbb{Q} | x \neq 1\}$.

Exercise 5 *Calculate the composition $f \circ g$ and $g \circ f$ for the functions f and g given by the following rules:*

$f : n \mapsto n - 2 \qquad g : n \mapsto 2n + 3$

$f : n \mapsto \frac{1}{n-1} \qquad g : n \mapsto n^2$

$f : n \mapsto n^2 - 1 \qquad g : n \mapsto 3n$

Exercise 6 *Determine the natural domain for the functions given by the following rules:*

$n \mapsto n - 2$

$n \mapsto \frac{n+1}{(3n-1)(n-2)}$

$n \mapsto \frac{3}{(n^2+1)n}$

$n \mapsto \sqrt{n - 2}$

$n \mapsto \frac{\sqrt{n+1}}{1-n^2}$

$n \mapsto \sqrt{2n}$