Lesson 3 - Fractional and literal equations

Fractional equations

We consider now equations in which the variable appears in the denominator of at least one fraction on one of its sides. These equations are called **fractional equations**.

For example

$$2x + \frac{5}{3x} = \frac{4x-3}{2} - 3$$

is a fractional equation.

Existence conditions of an equation

When we are given a fractional equation it is possible that, for certain values assigned to the variables, some algebraic expressions become meaningless.

It is therefore necessary to specify the **existence conditions (E.C.)** under which the equation is properly defined. In particular we must assure that each denominator of every fraction does not become 0 after substitution of numeric values to the variables.

Example 1 *Consider the equation*

$$\frac{1}{x-1} + \frac{2}{x-1} = 7$$

It is clear that we cannot assign the value 1 to the variable x because the expression $x - 1$ would become 0. So the existence conditions are given by $E.C.: \quad x \neq 1$.

Exercise 1 *Find the existence conditions of the equation*

$$\frac{x-1}{x+3} - \frac{2}{x} = \frac{5}{x-1}$$

Solution:

The denominators appearing in the equation are $x + 3$, x, $x - 1$ and these become 0 for the values -3, 0 and 1 respectively. So the existence conditions are given by $E.C.: \quad x \neq -3 \wedge x \neq 0 \wedge x \neq 1$.

Solution of a fractional equation

To solve a fractional equation we first determine the existence conditions of the equation. Then we try to get rid of the denominators applying the equivalence principle of multiplication/division to obtain an equation of first-degree which we know how to solve. The solution is *acceptable* only if it is *compatible* with the existence conditions.

Example 2 *Solve the equation*

$$\frac{x}{x-1} = \frac{1}{x-1}$$

Solution:

The existence conditions are easily seen to be E.C.: $x \neq 1$.

Now we can multiply both terms of the equation by $x - 1$ which is supposed to be non zero. We obtain

$$\frac{x}{x-1}(x-1) = \frac{1}{x-1}(x-1)$$

and after simplification we get the solution $x = 1$. But this is incompatible with the E.C. so the equation is impossible.

Exercise 2 *Solve the equation*

$$\frac{x-1}{x+5}-4=0$$

Solution:

Exercise 3 *Solve the equation*

$$x+\frac{4}{4-x}=\frac{x}{4-x}+x+4$$

Solution:

Exercise 4 *Solve the equation*

$$\frac{x-1}{x^2+3x}+\frac{2}{x}+\frac{9}{2x+6}=0$$

Solution:

Literal equations

We consider now equations in which there is more than one variable. For example, the equation

$$ax + 2 - 4x = b$$

contains the variables a, b, x. To solve this equation for x means to express x in terms of the other variables, which we call **parameters**. In our case the parameters are a and b.

These equations are called **literal equations**.

Solution of a literal equation

To solve a literal equation it is necessary to discuss for which values of the letters that we consider as parameters, the equation is *determined*, *undetermined* or *impossible*. In practice we first use the same principles seen in the preceding lessons to transform the equation in canonical form.

Example 3 *Solve the equation* $ax - 2a = 5x$ *for* x.

Solution:

We move all terms with x to the left side and all terms not containing x to the right. We obtain the equation

$$ax - 5x = 2a$$

We collect similar term and get

$$(a-5)x = 2a$$

Now we are ready for the discussion. In fact, to get the solution we can divide both terms by $a - 5$ but this is possible only if $a - 5 \neq 0$ or $a \neq 5$.

So, we distinguish two cases

1. $\boxed{a \neq 5}$

In this case the equation is *determined* and its solution is $x = \dfrac{2a}{a-5}$

2. $\boxed{a = 5}$

In this case we substitute the value 5 in place of a into the equation and get the relation $0x = 2 \cdot 5$ or $0x = 10$. This is an *impossible* equation.

Exercise 5 *Solve the equation* $ax + 2 = 2a + 6 - 2x$ *for* x.

Solution:

We move all terms with x to the left and all other terms to the right and obtain (after simplification) the equation

$$(a+2)x = 2a + 4$$

We distinguish the cases:

1. $a \neq -2$

In this case $a + 2 \neq 0$ and we can divide both terms by $a + 2$ to get the solution $x = \dfrac{2a+4}{a+2} = \dfrac{2(a+2)}{a+2} = 2$. The equation is *determined*.

2. $a = -2$

In this case $a + 2 = 0$ and we the equation becomes $0x = 0$. This is an *undetermined* equation.

Exercise 6 *Solve the equation* $(a+1)x = 3$ *for* x.

Solution:

Exercise 7 *Solve the equation* $2x - 4(3x-3) = 6(a-2x) + 6a$ *for* x.

Solution:

Exercise 8 *Solve the equation* $a(a-2)x = 2a + a^2$ *for* x.