Lesson 2 - First-degree equations

First-degree equations

We consider now equations whose left and right sides are defined by polynomials in a single variable x. This means that x is allowed to appear only in the numerator of fractions. If an equation of such a type can be put in normal form $ax + b = 0$ where a, b are numbers, then it is called a **first-degree** equation, because the reduced polynomial in its left hand side has degree one. A first-degree equation is also called a **linear equation**.

For example

$$2x + \frac{5}{3} = 0$$

is a first-degree equation.

Remark

We allow the variable x appearing only in the numerator. For example the equation

$$\frac{1}{x} - 3 = 0$$

is not a first-degree equation.

Canonical form

By applying the principles of addition/subtraction and multiplication/division seen in the previous lesson, a first-degree equation can always be transformed in the form

$$ax = b$$

with $a, b \in \mathbb{R}$. This is called the **canonical form** of the equation.

Solution of a first-degree equation

We now consider a first-degree equation in canonical form $ax = b$, where $a, b \in \mathbb{R}$.

Determined equations

If $a \neq 0$ we can apply the multiplication/division principle and divide both expression on the left and right side by a to obtain the equivalent equation

$$x = \frac{b}{a}$$

This means that we have *solved* the equation. Its *only solution* is $x = \frac{b}{a}$; the solution is unique because the result of division by a is unique. We call this a **determined equation**.

Undetermined equations

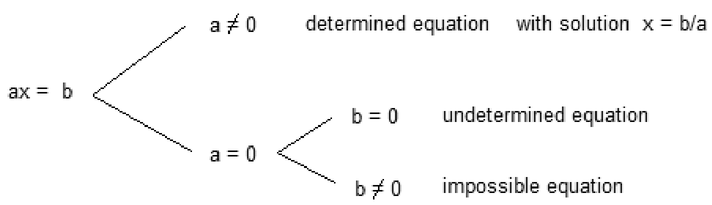
But what happens if $a = 0$? In this case we have to distinguish between two other cases, that is the case $b = 0$ and the case $b \neq 0$.

If $b = 0$ then the equation becomes $0x = 0$ (because both a and b are 0). We note that *every number* substituted in place of the variable x satisfies the equation and so the equation has an infinite number of solutions. We call this an **undetermined equation**.

Impossible equations

If $a = 0$ and $b \neq 0$ the situation is completely different. The equation has now the form $0x = b$ where $b \neq 0$ and it is clear that it has *no solution* because every number multiplied by 0 gives 0 and so the product $0 \cdot x$ cannot be equal to the non-zero number b. The solution set of this kind of equation is empty, so the equation is **impossible**.

We can summarize the preceding discussion in the following diagram, giving the complete theory of first-degree equations in one variable.



Strategy

The main idea to solve a first-degree equation is to transform it in canonical form. The procedure involves the application of the principles and rules seen in the previous lesson. We can summarize the steps as follows.

1. do all algebraic operations in both sides of the equation
2. cancel equal terms appearing in both sides (Cancellation rule)
3. move each term containing the variable from the right to the left side changing its sign
4. move each constant term from the left to the right side changing its sign
5. collect like terms in both sides and get the canonical form $ax = b$

Example 1 *Solve the equation* $4x - 9 + (x - 1)(x + 1) = (x - 3)^2 + 2x + 5$

First we do all the calculations (special products, products, ...) and we get the following equivalent equation

$$4x - 9 + \cancel{x^2} - 1 = \cancel{x^2} + 9 - 6x + 2x + 5$$

Then we cancel equal terms appearing on both sides and we obtain

$$4x - 9 - 1 = 9 - 6x + 2x + 5$$

We now collect like terms and get the equation

$$4x - 10 = 14 - 4x$$

We can now move all terms containing the variable x from the right to the left side and all constant terms from the left to the right side changing their signs

$$4x + 4x = 14 + 10$$

We finally collect equal terms and get the equation in canonical form

$$8x = 24$$

We obtain the solution dividing both terms by 8 (the coefficient of x). So $x = 3$ is the solution required.□

Example 2 *Solve the equation* $\dfrac{x}{2} = x + \dfrac{5}{3}$

We can write all terms under a common denominator. The l.c.m. of all denominators appearing on both sides is 6 so we can write the equivalent equation

$$\frac{3x}{6} = \frac{6x + 10}{6}$$

Applying the multiplication principle we can get rid of the denominators by simply multiplying both terms by 6. We get the equivalent equation

$$3x = 6x + 10$$

The solution can now be found as was done in the previous example and we leave it as an exercise. □

Exercises

Exercise 1 *Consider the following four equations:*
1. $0 = 2x + 1$ *2.* $x = x + x + 1$ *3.* $x = x + 2$ *4.* $2x = x + 1$
Which are impossible?

□ *First and second* □ *Second and fourth* □ *Only the first* □ *Only the second* □ *Only the third*

Exercise 2 *Consider the following four equations:*
1. $x + 1 = 2x + 2$ *2.* $2x + 1 = 2x + 1$ *3.* $x + 1 = 2x + 1$ *4.* $2x + 2 = 2x + 1$
Which are undetermined?

□ *First and third* □ *Second and first* □ *Third and fourth* □ *Fourth and first* □ *Only the second*

Exercise 3 *Solve the following equation*

$$(3x - 1)^2 + 2x(1 - x) + 2 = x - 7(1 - x)x$$

Solution:

Exercise 4 *Solve the following equation*

$$3\left(\frac{1}{2}x-1\right)-(1+x)+\frac{1}{3}\left(2x+\frac{1}{2}\right)=\frac{1}{2}x+1$$

Solution:

Exercise 5 *Solve the following equation*

$$\frac{x+1}{3}-\frac{2(x+1)}{5}+\frac{2}{3}=\frac{x-4}{5}-\frac{4}{15}x$$

Solution: