Lesson 1 - Identities and Equations

What is an equation? Some terminology and definitions

An **equation** is a mathematical sentence that states that two algebraic expressions are equal. The two expressions are separated by an equal sign.

For example

$$x + 5 = 7$$

is an equation. The expression $x+5$ is called the *left side*, or *left member*, and 7 the *right side*, or *right member*, of the equation.

An equation can be a *true sentence* such as $16 - 4 = 12$; it can be a *false sentence* such as $3 - 4 = 7$ or it can be an *open sentence* such as $x + 5 = 7$.

As in this last case, one or both of the expressions may contain variables. *Solving an equation* means finding **all** the values of the variables that make the sentence true.

The values are taken from a specified set of numbers, called the **domain** of the equation. We will consider mainly the set of rational numbers $\mathbb{Q}$ and the set of real numbers $\mathbb{R}$.

Each assignment of values to the variables that makes the sentence true is called a **solution** of the equation. The set of all solutions is called the **solution set** of the equation.

In the example above, 2 is a solution of the equation $x + 5 = 2$, because if we replace the variable x by 2 we obtain the true sentence $2 + 5 = 7$.

We say that two equations are **equivalent** if they have the same solution set.

We say that an equation is **impossible** if the solution set is the empty set $\emptyset$.

If every element of the domain is a solution of the equation, the equation is an **identity**.

Remark

The solutions of an equation depend on the domain considered. For example the equation

$$2x + 1 = 2$$

has no solution in the set of integers $\mathbb{Z}$, because for every integer value of x the number $2x + 1$ is odd and so cannot be equal to 2. But it has a solution in the set of rational numbers: $\frac{1}{2}$ is a solution, as can be easily verified.

Solving an equation: equivalence principles

To *solve* an equation the main idea is to transform both sides of the equation to obtain an equivalent equation which is possibly simpler to solve and obviously has the same solution set as the original equation.

The transformation procedure lies on some basic properties of equality.

Properties of equality

1. **Addition/subtraction property**: if equal expressions are added or subtracted to equal expressions the resulting sums and differences are equal

2. **Multiplication/division property**: if equal expressions are multiplied or divided by nonzero equal expressions the resulting products and quotients are equal

Example 1 *We want to solve the equation* $x + 5 = 7$.

We can apply the addition/subtraction property, adding the number -5 to both sides of the equation to obtain the following three equivalent equations

$$x + 5 - 5 = 7 - 5$$

$$x + 0 = 2$$

$$x = 2$$

In the last equation the variable x is alone on one side and it is easy to read the solution, $x = 2$.

Example 2 *We want to solve the equation* $2x + 3 = 9$.

As in the previous example, we can first add the value -3 to both sides of the equation to obtain the equivalent equation $2x = 6$.

We can now apply the multiplication/division property, multiplying by $\frac{1}{2}$ both sides of the new equation.

$$\frac{1}{2} \cdot 2x = \frac{1}{2} \cdot 6$$

$$\left(\frac{1}{2} \cdot 2\right) x = 3$$

$$x = 3$$

In the last equation the variable x is alone on one side and it is easy to read the solution, $x = 3$.

Solving an equation: techniques and some useful rules

An equation is not always written in a simple form. Before starting to solve the equation we should try to simplify both sides of the equation, for example *removing parentheses* and *collecting like terms*. These algebraic manipulations do not alter the solutions of the original equation.

Example 3 *We want to simplify the equation*

$$2x + 3 - (5x - 4) + 7x = 9 - x + 3 - (8 - 3x)$$

As a first step we can remove parentheses, paying attention to algebraic signs, to obtain the equivalent equation

$$2x + 3 - 5x + 4 + 7x = 9 - x + 3 - 8 + 3x$$

Then we can collect like terms to obtain

$$4x + 7 = 2x + 4$$

which has a simpler form than the original equation.

Some useful rules

As a direct consequence of the properties of addition/subtraction and multiplication/division we derive two useful rules for solving an equation.

Rule 1: **Transpose**

From a given equation we obtain an equivalent one if we **move** one term from one side to the other, **changing its sign**.

Example 4 *Consider the equations*

$$(a)\quad 9x + 4 = 5x + 7 \qquad\qquad (b)\quad 9x = 5x + 7 - 4$$

Equation (b) has been obtained from equation (a) by moving the term 4 from the left side to the right side changing its sign.

Example 5 *Consider the equations*

$$(a)\quad 2x - 3 = 6x + 7 \qquad\qquad (b)\quad 2x - 3 - 6x = 7$$

Equation (b) has been obtained from equation (a) by moving the term $6x$ from the right side to the left side changing its sign.

Rule 2: **Cancel**

From a given equation we obtain an equivalent one if we **cancel** an equal term from both sides of the equation.

Example 6 *Consider the equations*

$$(a)\quad 9x + \cancel{4} - 2x = 5x + \cancel{4} + 8 \qquad\qquad (b)\quad 9x - 2x = 5x + 8$$

Equation (b) has been obtained from equation (a) cancelling the term 4 that appears on both sides.

Example 7 *Consider the equations*

$$(a)\quad 2x - \cancel{3x} + 9 = 6x + 7 - \cancel{3x} \qquad\qquad (b)\quad 2x + 9 = 6x + 7$$

Equation (b) has been obtained from equation (a) cancelling the term $-3x$ that appears on both sides.

Normal form of an equation

An equation is in **normal form** if it is written as $P = 0$ where P is a polynomial whose terms are not similar.

For example $5x^3 - 2x^2 + 3 = 0$ is in normal form but $5x^3 - x + 2x^2 + 5x = 0$ is not.

The **degree** of an equation $P = 0$ in normal form is the degree of the polynomial P. To compute the degree of an arbitrary equation we must first transform it in normal form.

Examples

a) The equations $5x + 4 = 0$ and $x^3 - 3x^2 + 5x - 7 = 0$ are respectively of first and third degree.

b) The equation $3x^2 + 5x - 1 - 3x - 2x^2 - x^2 = 0$ is **not** of second degree. In fact, by collecting like terms, we obtain the equivalent equation in normal form $2x - 1 = 0$. This shows that the equation is of first degree.

Worked Exercises

Exercise 1 *Check if* $x = -9$ *is a solution for the equation* $x + 6 = -3$.

Solution: Remembering the concept of solution of an equation, we have to substitute the value -9 in place of x in the equation and verify if the resulting numeric identity is true or false. In this case we obtain $-9 + 6 = -3$ which is true. So, -9 is a solution.

Exercise 2 *Try to solve the equation* $11 + 3x - 7 = 6x + 5 - 3x$ *by first simplifying it and then applying the addition/subtraction principle.*

Solution: We can write the following equivalent equations:

$11 + 3x - 7 = 6x + 5 - 3x$ original equation

$4 + 3x = 3x + 5$ collecting like terms in both sides

$4 + 3x - 3x = 3x + 5 - 3x$ adding the term $-3x$ to both sides

$4 = 5$ collecting like terms in both sides

We ended up with a *false* numerical sentence. So the equation has **no solution**.

Exercise 3 *Try to solve the equation* $6x + 5 - 2x = 4 + 4x + 1$ *by first simplifying it and then applying the addition/subtraction principle.*

Solution: We can write the following equivalent equations:

$6x + 5 - 2x = 4 + 4x + 1$ original equation

$4x + 5 = 4x + 5$ collecting like terms in both sides

$4x + 5 - 4x = 4x + 5 - 4x$ adding the term $-4x$ to both sides

$5 = 5$ collecting like terms in both sides

We ended up with an always *true* numerical sentence. So every value is a solution and the equation is **an identity**.

Exercise 4 *Put the equation* $(2x - 3)^2 - x(4x^2 + 3) - 9 = 0$ *in normal form and determine its degree.*

Solution: We do the algebraic calculations as follows:

$4x^2 + 9 - 12x - 4x^3 - 3x - 9 = 0$ special products and multiplication

$-4x^3 + 4x^2 - 15x = 0$ collecting like terms

The equation has now the form $P = 0$ where $P = -4x^3 + 4x^2 - 15x$ is a reduced polynomial of third degree. So the equation is of third degree.

Exercise 5 *The following equations are all equivalent to* $x = -2$*, but one. Find it.*

a) $x+1 = 2x+3$ *b)* $-x = 2$ *c)* $2x+1 = -3$ *d)* $2x+3 = 0$ *e)* $3(x+2) = 2(x+2)$

Solution: We remember the concept of equivalent equations: two equations are equivalent if they have the same solutions. Clearly $x = -2$ has -2 as solution and so we have to find the equation that does not have -2 as solution.

We easily check that equation *d)* doesn't have -2 as solution: in fact $2(-2)+3 = 0$ is a false numerical statement. We leave all the other calculations to the reader.